EXERCISE SET 2.2

Computation

1. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & 4 \\ 2 & 1 & 3 \end{bmatrix}$,

$C = \begin{bmatrix} 2 & 3 \\ 6 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$. Calculate, if possible

*(a) AB and BA (b) AC and CA

*(c) AD and DA.

Observe that $AB \neq BA$ since BA does not exist, $AC \neq CA$ and $AD = DA$, illustrating the different possibilities when order is reversed in matrix multiplication.

***2.** Compute $A(BC)$ and $(AB)C$ for the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Observe that these products are equal, illustrating the associative property of matrix multiplication.

3. Compute the product ABC for the following three matrices in two distinct ways.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$$

4. Compute each of the following linear combinations for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 0 \\ 3 & 4 \end{bmatrix}$.

*(a) $2A + 3B$ (b) $A + 2B + 4C$

*(c) $3A + B - 2C$

5. Compute each of the following expressions for $A = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$.

*(a) $(AB)^2$ (b) $A - 3B^2$

*(c) $A^2B + 2C^3$ (d) $2A^2 - 2A + 3I_2$

Sizes of Matrix Products

6. Given that A is a 4×2 matrix, B is 2×6, C is 3×4, and D is 6×3, determine the sizes of the following products, if they exist.

*(a) ABC (b) ABD *(c) CAB

(d) $DCAB$ (e) A^2BDC

7. If P is 3×2, Q is 2×1, R is 1×3, S is 3×1, and T is 3×3, determine the sizes of the following matrix expressions, if they exist.

*(a) PQR (b) $PQ + TPQ$

*(c) $5QR - 2TPR$ (d) $4SPQ + 3PQ$

*(e) $QRSR + QR$

Matrix Operations

8. Let A be an $m \times n$ matrix. Prove that AB and BA both exist if and only if B is an $n \times m$ matrix.

9. Verify the following properties of matrix operations given in this section:

*(a) the associative property of matrix addition $A + (B + C) = (A + B) + C$

(b) the distributive property $c(A + B) = cA + cB$

(c) $AI_n = I_nA = A$ if A is an $n \times n$ matrix.

10. Let A be an $m \times n$ matrix. Show that $AI_n = A$.

***11.** Let A be any $m \times n$ matrix, O_{mn} be the $m \times n$ zero matrix, and c be a scalar. Show that if $cA = O_{mn}$ then either $c = 0$ or $A = O_{mn}$.

12. Simplify the following matrix expressions.

*(a) $A(A - 4B) + 2B(A + B) - A^2 + 7B^2 + 3AB$

(b) $B(2I_n - BA) + B(4I_n + 5A)B - 3BAB + 7B$

*(c) $(A - B)(A + B) - (A + B)^2$

13. Simplify the following matrix expressions.

*(a) $A(A + B) - B(A + B)$

(b) $A(A - B)B + B2AB - 3A^2$

*(c) $(A + B)^3 - 2A^3 - 3ABA - A3B^2 - B^3$

14. Find all the matrices that commute with the following matrices.

*(a) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

***15.** What is incorrect about the following proof? Let $AX = B$ be a system of linear equations with solutions X_1 and X_2. Then

$$AX_1 = B \quad \text{and} \quad AX_2 = B$$
$$AX_1 = AX_2$$
$$X_1 = X_2$$

Thus every system of linear equations has at most one solution.

Powers of Matrices

16.*(a) Let A be an $n \times n$ matrix. Prove that A^2 is an $n \times n$ matrix.

(b) Let A be an $m \times n$ matrix, with $m \neq n$. Prove that A^2 does not exist.

Thus one can only talk about powers of square matrices.

***17.** If A and B are square matrices of the same size, prove that in general

$$(A + B)^2 \neq A^2 + 2AB + B^2$$

Under what condition does equality hold?

18. If A and B are square matrices of the same size such that $AB = BA$, prove that $(AB)^2 = A^2B^2$. By constructing an example, show that this result does not hold for all square matrices of the same size.

19. If n is a nonnegative integer and A and B are square matrices of the same size such that $AB = BA$, prove that $(AB)^n = A^nB^n$. By constructing an example, show that this identity does not hold in general for all square matrices of the same size.

20. Show that nonnegative integer powers of the same matrix commute. ($A^rA^s = A^sA^r$)

Diagonal Matrices

21. Let A and B be diagonal matrices of the same size and c a scalar. Prove that *(a) $A + B$ is diagonal (b) cA is diagonal (c) AB is diagonal.

22. If A and B are diagonal matrices of the same size, prove that $AB = BA$.

23. Prove that if a matrix A commutes with a diagonal matrix which has no two diagonal elements the same, then A is a diagonal matrix.

Idempotent and Nilpotent Matrices

A square matrix A is said to be **idempotent** if $A^2 = A$. A square matrix A is said to be **nilpotent** if there is a positive integer p such that $A^p = 0$. The least integer such that $A^p = 0$ is called the **degree of nilpotency** of the matrix.

24. Determine whether the following matrices are idempotent.

*(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

*(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$

*(e) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

25. Determine b, c, and d such that $\begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$ is idempotent.

26. Determine a, c, and d such that $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ is idempotent.

27. Prove that if A and B are idempotent and $AB = BA$, then AB is idempotent.

28. Show that if A is idempotent, and if n is a positive integer, then $A^n = A$.

29. Show that the following matrices are nilpotent with degree of nilpotency 2.

(a) $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & 8 \\ -2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}$

30. Show that the following matrix is nilpotent with degree of nilpotency 3.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Systems of Linear Equations

31. Write each of the following systems of linear equations as a single matrix equation $AX = B$.

*(a) $\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 3x_1 - 8x_2 &= -1 \end{aligned}$ (b) $\begin{aligned} 4x_1 + 7x_2 &= -2 \\ -2x_1 + 3x_2 &= -4 \end{aligned}$

(c) $\begin{aligned} -9x_1 - 3x_2 &= -4 \\ 6x_1 - 2x_2 &= 7 \end{aligned}$

32. Write each of the following systems of linear equations as a single matrix equation $AX = B$.

*(a) $\begin{aligned} x_1 + 8x_2 - 2x_3 &= 3 \\ 4x_1 - 7x_2 + x_3 &= -3 \\ -2x_1 - 5x_2 - 2x_3 &= 1 \end{aligned}$

*(b) $\begin{aligned} 5x_1 + 2x_2 &= 6 \\ 4x_1 - 3x_2 &= -2 \\ 3x_1 + x_2 &= 9 \end{aligned}$

(c) $\begin{aligned} x_1 - 3x_2 + 6x_3 &= 2 \\ 7x_1 + 5x_2 + x_3 &= -9 \end{aligned}$

(d) $\begin{aligned} 2x_1 + 5x_2 - 3x_3 + 4x_4 &= 4 \\ x_1 + 9x_3 + 5x_4 &= 12 \\ 3x_1 - 3x_2 - 8x_3 + 5x_4 &= -2 \end{aligned}$

*33. Prove that if X_1 and X_2 are solutions to the homogeneous system of linear equations $AX = 0$, then the linear combination $aX_1 + bX_2$ is a solution for all scalars a and b.

*34. Consider the following system of equations. You are given two solutions, X_1 and X_2. Generate four other solutions using the operations of addition and scalar multiplication. Use the result of Exercise 33 to find a solution for which $x_1 = 1$ and $x_2 = 0$.

$$\begin{aligned} x_1 + 2x_2 - x_3 - 2x_4 &= 0 \\ 2x_1 + 5x_2 - 2x_4 &= 0 \\ 4x_1 + 9x_2 - 2x_3 - 6x_4 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \end{aligned}$$

$$X_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 11 \\ -4 \\ 1 \\ 1 \end{bmatrix}$$

35. Consider the following system of four equations. You are given two solutions X_1 and X_2. Generate four other solutions using the operations of addition and scalar multiplication. Find a solution for which $x_1 = 6$ and $x_2 = 9$.

$$\begin{aligned} x_1 - x_2 - x_3 + 2x_4 &= 0 \\ 3x_1 - 2x_2 + x_3 + 3x_4 &= 0 \end{aligned}$$

continued